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A SOLUTION PROCEDURE FOR THREE-DIMENSIONAL INCOMPRESSIBLE NAVIER-STOKES EQUATION AND ITS APPLICATION

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I. Introduction

A major difficulty when solving the incompressible flow equations that use primitive variables is caused by the pressure term which is used as a mapping parameter to obtain a divergence-free velocity field. One commonly used approach is to solve the Poisson equation for pressure, which is derived from the momentum equations [1]. This approach can be very time consuming. To accelerate the pressure-field solution and alleviate the drawback associated with the Poisson equation approach, Chorin [2] proposed the use of artificial compressibility in solving the continuity equation. A similar method was adopted by Steger and Kutler [3] and Chakravarthy [4] using an implicit approximate-factorization scheme [5]. Based on this procedure, a pseudocompressible method has been developed for solving three-dimensional, viscous, incompressible flow problems cast in generalized curvilinear coordinates [6,7]. The purpose of the present paper is to show salient features of the pseudocompressible approach, which is primarily designed for obtaining steady-state solutions efficiently.

II. Description of the Method

In the present formulation, the three-dimensional, incompressible Navier-Stokes equations are modified to form the following set of governing equations written in dimensionless form :

$$\frac{1}{\beta}\frac{\partial p}{\partial t}+\frac{\partial u_i}{\partial x_i}=0 \tag{1a}$$

$$\frac{\partial u_i}{\partial t}+\frac{\partial u_i u_j}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\frac{\partial \tau_{ij}}{\partial x_j} \tag{1b}$$

Here, t is time; x_i are the Cartesian coordinates; u_i are corresponding velocity components; p is the pressure; and τ_{ij} is the viscous stress tensor. The parameter $1/\beta$ is the pseudocompressibility. As the solution converges to a steady state, the pseudocompressibility effect approaches zero, yielding the incompressible form of the equations. In the present study, the approximate factorization scheme by Beam and Warming [5] is implemented to solve the finite-difference form of the governing equations written in general curvilinear coordinates (see ref. 6 for detail).

In the present formulation, waves of finite speed are introduced. And the system of modified equations given by equations (1a) and (1b) can be marched in time. The magnitude of the wave speed depends on β. To recover the incompressible phenomena, the physics requires that the pressure wave propagates much faster than the spreading of vorticity. From this, the following criterion for the lower bound on β is obtained [7]:

$$\beta > [1 + 4(x_{ref}/x_\delta)^2(x_L/x_{ref})/Re]^2 - 1 \qquad (2)$$

where x_{ref} is the reference length, and x_δ and x_L are the characteristic lengths that the vorticity and the pressure waves have to propagate during a given timespan.

The upper bound on β depends upon the particular numerical algorithm chosen. In the present study, higher-order cross-differencing terms are added to obtain the approximately factored form of the governing equations. These added terms contaminate the momentum equations as well as the continuity equation, and therefore must be kept smaller than the original terms everywhere in the computational domain. This requirement leads to the following criterion for the upper bound of β:

$$\beta\,\Delta\tau < O(1) \qquad (3)$$

where $\Delta\tau$ is the time-step used in the integration scheme.

III. Computed Results

Numerical experiments were performed to illustrate the present procedure. To represent an internal flow, the flow through a channel at Re=1,000 was chosen. The coordinate system and velocity vectors for a converged solution are shown in figures 1a and 1b. To change the ratio of the time scales required for the pressure waves and the vorticity to map the entire flow field, the channel length, L, is varied form 20 to 40. The recommended values of β for these cases using $\Delta\tau = 0.1$ are:

$$0.75 < \beta_{L=20} < 10, \quad 1.19 < \beta_{L=30} < 10, \quad 1.69 < \beta_{L=40} < 10$$

In table 1, the number of iterations for one roundtrip by the pressure wave (denoted by N_1) is tabulated for various values of β which include values outside the recommended range. In figure 2, root-mean-square (RMS) values of $(div\, u)$ are plotted to check the accuracy of the converged solutions. When the value of β is out of the range specified, the accuracy of the solution deteriorates.

To represent an external flow, the flow past a circular cylinder at a $Re = 40$ was chosen. To obtain the near-field solution only, the distance traveled by the waves and the spreading of the vorticity can be approximately the same in magnitude. In the present case, this leads to the range for β using $\Delta\tau = 0.1$ to be $0.1 < \beta < 10$. This indicates that the magnitude of β is less restrictive for external flows. In figures 3a and 3b, the stream-function contours and the pressure coefficient on the surface are shown for a steady-state solution. This solution agrees very well with that of Mehta who used a stream function and vorticity formulation in two dimensions (private communication, U. B. Mehta, 1983). In figure 4, in which the history of the pressure drag is shown for an impulsively started circular cylinder at $Re = 40$, four different values of β were compared with the time-accurate solution of Mehta. In all cases, the values of β are selected within the suggested range above, and the solutions converge rapidly.

To test internal flows further, an annular duct with a 180° bend is chosen. This configuration is similar to the turnaround duct of the hot-gas manifold in the Space Shuttle main engine (SSME). In figures 5a and 5b, the geometry and a laminar solution at Re=1,000 are shown, which reveals the formation of a large separated bubble after the 180° bend. For this geometry, the streamwise length normalized by the duct width is 20. The test problems presented here were treated using a 51 x 17 x 21 mesh for half-duct formulation and the computing time required was 1.1 x 10^{-4} sec per mesh point per time-step on the Cray X-MP computer at NASA Ames Research Center.

IV. Concluding Remarks

This paper presents salient features of the computational procedure developed for a three-dimensional, incompressible, Navier-Stokes code. This procedure has been applied to various geometrically complex flows, including a major application in analyzing the flow field in the SSME power head. The present algorithm has been shown to be very robust and accurate if the selection of β is made according to the guidelines presented here.

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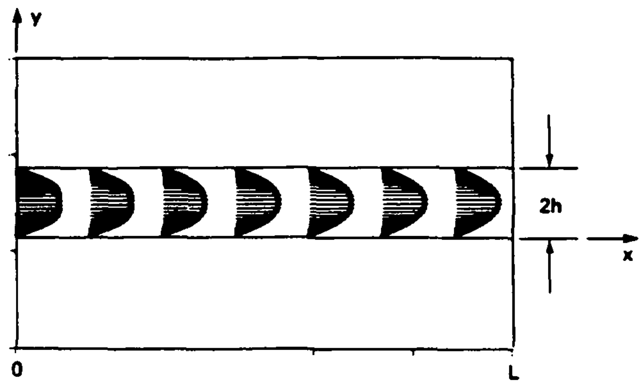
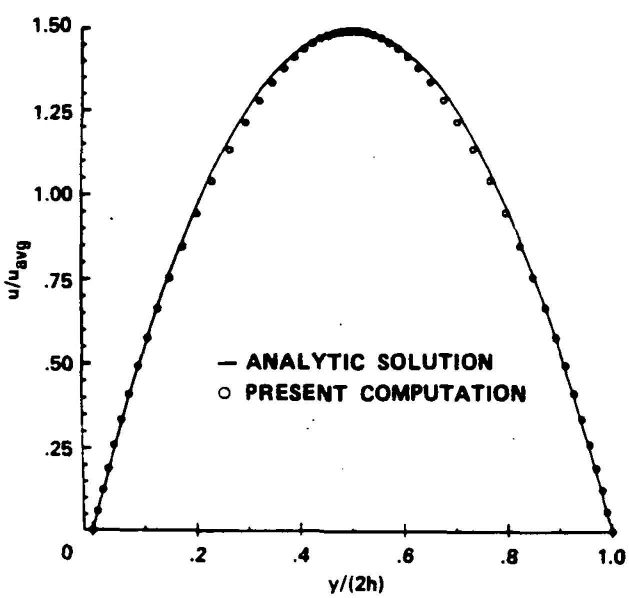
Table 1: Number of iterations required for one round-trip by pressure waves between in- and out-flow boundary of a channel: Re = 1000 and $\Delta\tau$ = 0.1

	β	0.1	1	2	5	10	50
N_1	L = 20	4196	566	347	196	133	58
	L = 30	6293	849	520	294	199	86
	L = 40	8391	1132	693	392	266	115

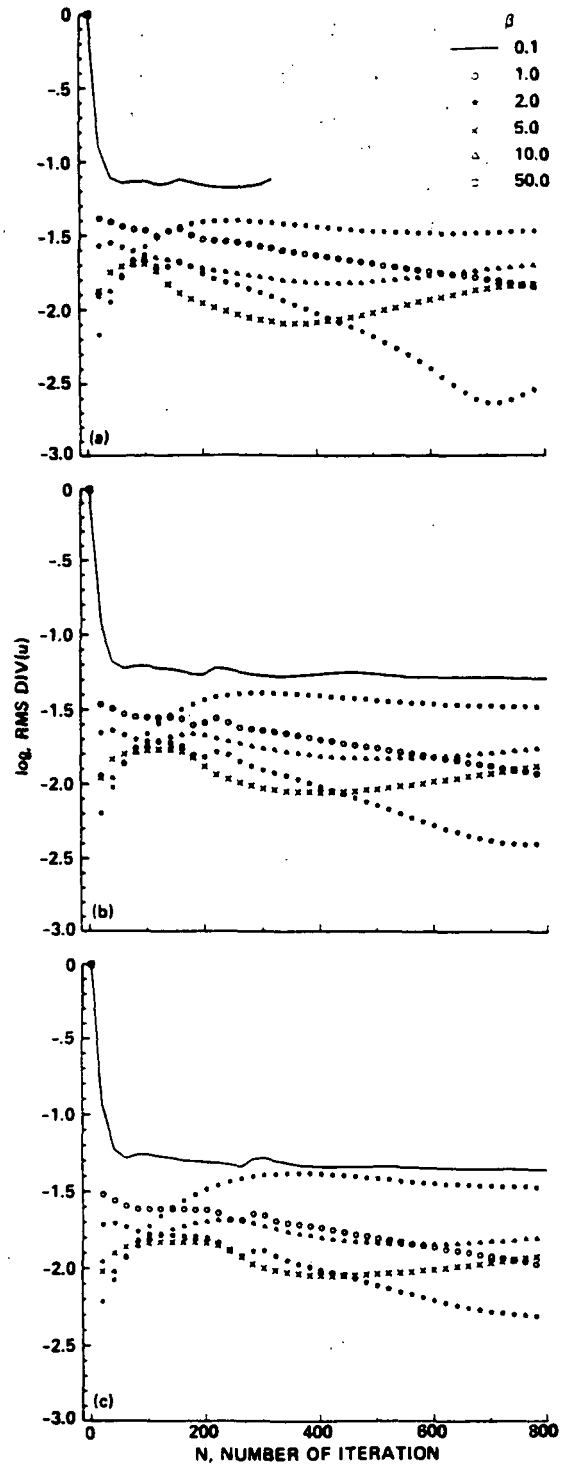


(a) Velocity vector



(b) Fully developed velocity profile

Figure 1.– Developing laminar channel flow at Re = 1,000 (Re based on channel width and average velocity).



(a) L = 20
(b) L = 30
(c) L = 30

Figure 2.– RMS (divu) history of channel flow at Re = 1,000 and $\Delta\tau$ = 0.1.

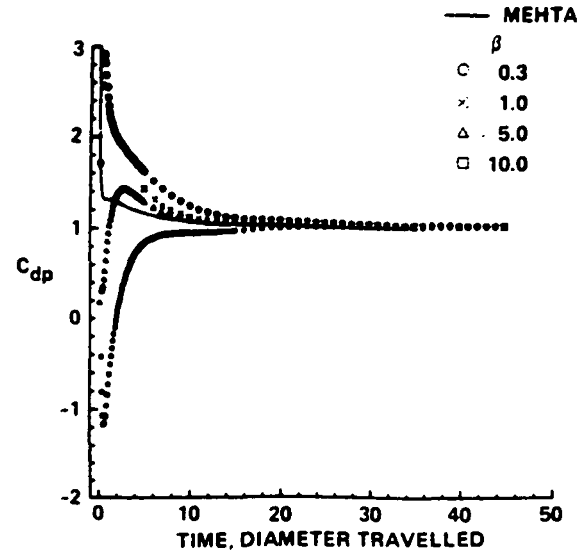
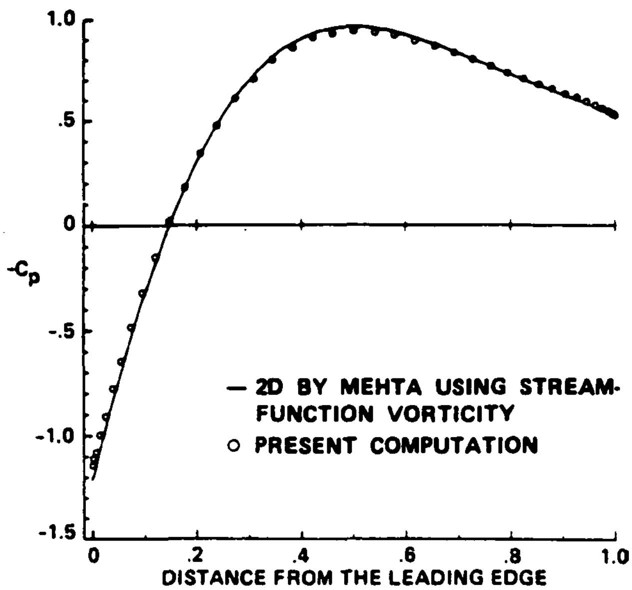
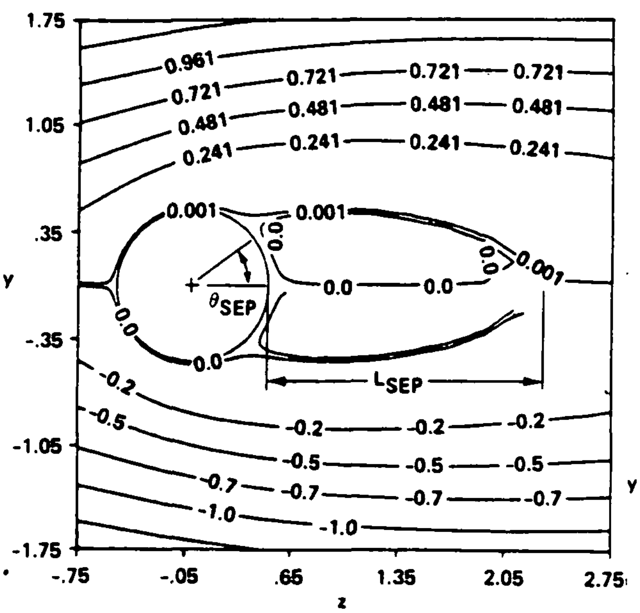


Figure 4.– Pressure drag history for flow over a circular cylinder at Re = 40.

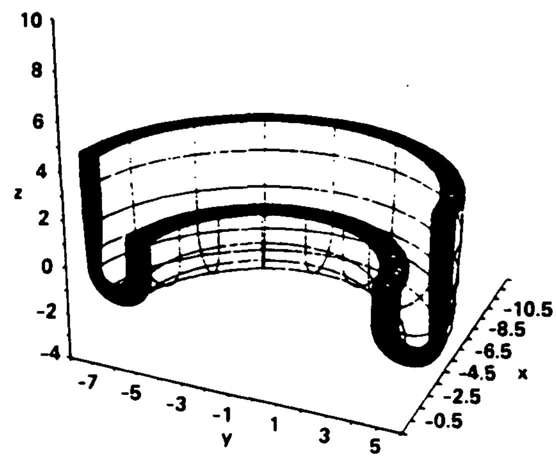


(a) Pressure coefficient on the surface

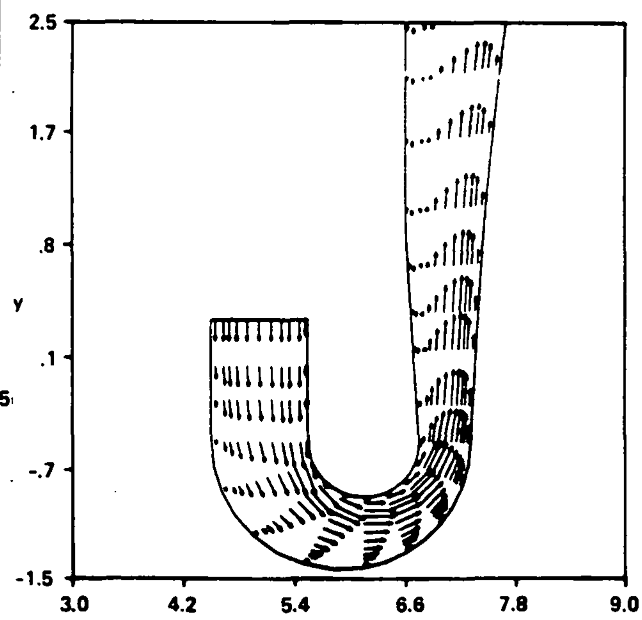


(b) Stream-function contours

Figure 3.– Steady-state solution for flow over a circular cylinder at Re = 40.



(a) Three-dimensional grid.



(b) Typical flow pattern with separation.

Figure 5.– Flow through a turnaround duct.

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16. Abstract

An implicit, finite-difference procedure is presented for numerically solving viscous incompressible flows. For convenience of applying the present method to three-dimensional problems, primitive variables, namely the pressure and velocities, are used. One of the major difficulties in solving incompressible flows that use primitive variables is caused by the pressure field solution method which is used as a mapping procedure to obtain a divergence-free velocity field. The present method is designed to accelerate the pressure-field solution procedure. This is achieved by the method of pseudocompressibility in which the time derivative pressure term is introduced into the mass conservation equation. The pressure wave propagation and the spreading of the viscous effect is investigated using simple test problems. The present study clarifies physical and numerical characteristics of the pseudo-compressible approach in simulating incompressible flows. Computed results for external and internal flows are presented to verify the present procedure. The present algorithm has been shown to be very robust and accurate if the selection of the pseudo-compressibility parameter, β, has been made according to the guidelines given in the paper.

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